

Global structure of radial positive solutions for a prescribed mean curvature problem in a ball

Ruyun Ma^{a,*} Hongliang Gao^b Yanqiong Lu^c

Department of Mathematics, Northwest Normal University, Lanzhou 730070, P R China

Abstract. In this paper, we are concerned with the global structure of radial positive solutions of boundary value problem

$$\operatorname{div}(\phi_N(\nabla v)) + \lambda f(|x|, v) = 0 \quad \text{in } B(R), \quad v = 0 \quad \text{on } \partial B(R),$$

where $\phi_N(y) = \frac{y}{\sqrt{1-|y|^2}}$, $y \in \mathbb{R}^N$, λ is a positive parameter, $B(R) = \{x \in \mathbb{R}^N : |x| < R\}$, and $|\cdot|$ denote the Euclidean norm in \mathbb{R}^N . All results, depending on the behavior of nonlinear term f near 0, are obtained by using global bifurcation techniques.

Keywords. Mean curvature operator; Minkowski space; Positive radial solutions; Bifurcation methods.

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1 Introduction

In this paper we are concerned with the global structure of radial positive solutions of Dirichlet problem in an ball, associated to mean curvature operator in flat Minkowski space

$$\mathbb{L}^{N+1} := \{(x, t) : x \in \mathbb{R}^N, t \in \mathbb{R}\}$$

endowed with the Lorentzian metric

$$\Sigma_{j=1}^N (dx_j)^2 - (dt)^2,$$

where (x, t) are the canonical coordinates in \mathbb{R}^{N+1} .

It is known (see e.g. [1, 4, 12, 28, 31]) that the study of spacelike submanifolds of codimension one in \mathbb{L}^{N+1} with prescribed mean extrinsic curvature leads to Dirichlet problems of the type

$$\mathcal{M}v = H(x, v) \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where

$$\mathcal{M}v = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right),$$

E-mail addresses: mary@nwnu.edu.cn(R.Ma), gaohongliang101@163.com(H.Gao), linmu8610@163.com(Y.Lu)

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Ω is a bounded domain in \mathbb{R}^N and the nonlinearity $H : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

The starting point of this type of problems is the seminal paper [12] which deals with entire solutions of $\mathcal{M}v = 0$. The equation $\mathcal{M}v = \text{constant}$ is then analyzed in [31], while $\mathcal{M}v = f(v)$ with a general nonlinearity f is considered in [9]. On the other hand, in [20] the author considered the Neumann problem

$$\mathcal{M}v = \kappa v + \lambda \quad \text{in} \quad B(R), \quad \partial_\nu v = 0 \quad \text{on} \quad \partial B(R),$$

where $B(R) = \{x \in \mathbb{R}^N : |x| < R\}$, $\lambda \neq 0$, $\kappa > 0$, $\mu \in [0, 1)$ and $N = 2$. More general sign changing nonlinearities are studied in [5].

If H is bounded, then it has been shown by Bartnik and Simon [4] that (1.1) has at least one solution $u \in C^1(\Omega) \cap W^{2,2}(\Omega)$. Also, when Ω is a ball or an annulus in \mathbb{R}^N and the nonlinearity H has a radial structure, then it has been proved in [6] that (1.1) has at least one classical radial solution. This can be seen as a *universal* existence result for the above problem in the radial case. On the other hand, in this context the existence of positive solutions has been scarcely explored in the related literature, see [7-8].

Very recently, Bereanu, Jebelean and Torres [7] used Leray-Schauder degree arguments and critical point theory for convex, lower semicontinuous perturbations of C^1 -functionals, proved existence of classical positive radial solutions for Dirichlet problems

$$\mathcal{M}v + f(|x|, v) = 0 \quad \text{in} \quad B(R), \quad v = 0 \quad \text{on} \quad \partial B(R), \quad (1.2)$$

under the condition

(H_1) $f : [0, R] \times [0, \alpha) \rightarrow \mathbb{R}$ is a continuous function, with $0 < \alpha \leq \infty$ and such that $f(r, s) > 0$ for all $(r, s) \in (0, R] \times (0, \alpha)$.

They proved the following

Theorem A [7, Theorem 1] Assume that (H_1) and $R < \alpha$ and

$$\lim_{s \rightarrow 0} \frac{f(r, s)}{s} = \infty \quad \text{uniformly for } r \in [0, R].$$

Then (1.2) has at least one positive radial solution.

Bereanu, Jebelean and Torres [8] used the upper and lower solutions and Leray-Schauder degree type arguments to study the special case of

$$\mathcal{M}v + \lambda \mu(|x|)v^q = 0 \quad \text{in} \quad B(R), \quad v = 0 \quad \text{on} \quad \partial B(R), \quad (1.3)$$

under the condition

(H_2) $N \geq 2$ is an integer, $R > 0$, $q > 1$ and $\mu : [0, \infty) \rightarrow \mathbb{R}$ is continuous, $\mu(r) > 0$ for all $r > 0$.

They proved the following

Theorem B [8, Theorem 1] Assume (H_2) holds. Then there exists $\Lambda > 2N/(\max_{[0,R]} \mu R^{q+1})$ such that problem (1.3) has zero, at least one or at least two positive solutions according to $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$ or $\lambda > \Lambda$. Moreover, Λ is strictly decreasing with respect to R .

Motivated by above papers, in this paper, we investigate the global structure of radial positive solutions of Dirichlet problem

$$\operatorname{div}(\phi_N(\nabla v)) + \lambda f(|x|, v) = 0 \quad \text{in } B(R), \quad v = 0 \quad \text{on } \partial B(R) \quad (1.4)$$

by the unilateral global bifurcation theory of [21, Sections 6.4, 6.5] and some preliminary results on the superior limit of a sequence of connected components due to Luo and Ma [24]. We shall make the following assumptions

(A1) $R \in (0, \infty)$ and $\delta \in [0, R)$, $f : [\delta, R] \times [0, \alpha) \rightarrow [0, \infty)$ is continuous for some $\alpha > R$, and $f(r, s) > 0$ for $(r, s) \in [\delta, R] \times (0, \alpha)$;

(A2) $\lim_{s \rightarrow 0^+} \frac{f(r, s)}{s} = m(r)$ uniformly $r \in [\delta, R]$ with $m \in C[\delta, R]$ is radially symmetric and $m(r) \geq 0, m(r) \not\equiv 0$ on any subinterval of $[\delta, R]$;

(A3) $\lim_{s \rightarrow 0^+} \frac{f(r, s)}{s} = \infty$ uniformly $r \in [\delta, R]$, and $f(r, 0) = 0$ for $r \in [\delta, R]$;

(A4) $\lim_{s \rightarrow 0^+} \frac{f(r, s)}{s} = 0$ uniformly $r \in [\delta, R]$.

Let $\phi_N(y) = \frac{y}{\sqrt{1-|y|^2}}$, $y \in \mathbb{R}^N$. Then by setting, as usual, $|x| = r$ and $v(x) = u(r)$, the problem (1.4) reduces to the mixed boundary value problem

$$(r^{N-1} \phi_1(u'))' + \lambda r^{N-1} f(r, u) = 0, \quad u'(\delta) = u(R) = 0 \quad (1.5)_\delta$$

with $\delta = 0$, where $\phi_1(s) = \frac{s}{\sqrt{1-s^2}}$, $s \in \mathbb{R}$.

To study the global structure of positive radial solutions of problem (1.4), we need to study the family of auxiliary problems $(1.5)_\delta$.

For given $\delta \in [0, R)$. Let

$$X_\delta = C[\delta, R], \quad E_\delta = \{u \in C^1[\delta, R] : u'(\delta) = u(R) = 0\}$$

be the Banach spaces endowed with the normals

$$\|u\|_{C[\delta, R]} = \sup_{r \in [\delta, R]} |u(r)|, \quad \|u\|_{C^1[\delta, R]} = \sup_{r \in [\delta, R]} |u(r)| + \sup_{r \in [\delta, R]} |u'(r)|,$$

respectively. Denoted by Σ_δ be the closure of the set

$$\{(\lambda, u) \in [0, \infty) \times C^1[\delta, R] : u \text{ satisfies } (1.5)_\delta, \text{ and } u \not\equiv 0\}$$

in $\mathbb{R} \times E_\delta$. Let

$$P_\delta = \{u \in E_\delta \mid u(t) \geq 0, t \in [\delta, R]\}.$$

Then P_δ is a positive cone of E_δ and $\text{int} P_\delta \neq \emptyset$. Let

$$P_\delta^0 = \{u \in X_\delta \mid u(t) \geq 0, t \in [\delta, R]\}.$$

Denoted by θ be the zero element in E_δ .

The main results of the paper are the following

Theorem 1.1. Let $\delta \in [0, R]$ be given and let $\lambda_1(m, \delta)$ be the principal eigenvalue of

$$-(r^{N-1}u')' = \lambda r^{N-1}m(r)u, \quad u'(\delta) = u(R) = 0. \quad (1.6)_\delta$$

Assume that (A1) and (A2) hold. Then there exists a connected component $\zeta \in \Sigma_\delta$, such that

- (a) $(\zeta \setminus \{(\lambda_1(m, \delta), \theta)\}) \subset ((0, \infty) \times \text{int} P_\delta)$;
- (b) ζ joins $(\lambda_1(m, \delta), \theta)$ with infinity in λ direction;
- (c) $\text{Proj}_{\mathbb{R}} \zeta = [\lambda_*, \infty) \subset (0, \infty)$;
- (d) for every $n \in \mathbb{N}$, $\lim_{(\lambda, u) \in \zeta, \lambda \rightarrow \infty} \text{meas}\{r \in [\delta, R] : |u'(r) - (-1)| > \frac{1}{n}\} = 0$;
- (e) $\lim_{(\lambda, u) \in \zeta, \lambda \rightarrow \infty} \|u\|_{C[\delta, R]} = R - \delta$.

Theorem 1.2 Let $\delta \in [0, R]$ be given. Assume that (A1) and (A3) hold. Then there exists a connected component $\zeta \in \Sigma_\delta$ such that

- (a) $(\zeta \setminus \{(0, \theta)\}) \subset ((0, \infty) \times \text{int} P_\delta)$;
- (b) ζ joins $(0, \theta)$ with infinity in λ direction;
- (c) $\text{Proj}_{\mathbb{R}} \zeta = [0, \infty)$;
- (d) for every $n \in \mathbb{N}$, $\lim_{(\lambda, u) \in \zeta, \lambda \rightarrow \infty} \text{meas}\{r \in [\delta, R] : |u'(r) - (-1)| > \frac{1}{n}\} = 0$;
- (e) $\lim_{(\lambda, u) \in \zeta, \lambda \rightarrow \infty} \|u\|_{C[\delta, R]} = R - \delta$.

Theorem 1.3 Let $\delta \in [0, R]$ be given. Assume that (A1) and (A4) hold. Then there exist a connected component $\zeta \in \Sigma_\delta$, such that

- (a) $\zeta \subset ((0, \infty) \times \text{int} P_\delta)$;
- (b) ζ joins (∞, θ) with $(\infty, R - \delta)$ in $\mathbb{R} \times X_\delta$;
- (c) there exists two constants $\Lambda > 0$ and $\rho_0 \in (0, R - \delta)$ such that

$$\zeta \cap \{(\mu, v) \in \Sigma_\delta \mid \mu \geq \Lambda, \|v\|_{C[\delta, R]} = \rho_0\} = \emptyset;$$

- (d) $\text{Proj}_{\mathbb{R}} \zeta = [\lambda_*, \infty) \subset (0, \infty)$;
- (e) for every $n \in \mathbb{N}$ and $(\lambda, u) \in \zeta$ with $\|u\|_{C[\delta, R]} \geq \rho_0$,

$$\lim_{\lambda \rightarrow \infty} \text{meas}\{r \in [\delta, R] : |u'(r) - (-1)| > \frac{1}{n}\} = 0;$$

(f) for $(\lambda, u) \in \zeta$ with $\|u\|_{C[\delta, R]} \geq \rho_0$,

$$\lim_{\lambda \rightarrow \infty} \|u\|_{C[\delta, R]} = R - \delta.$$

Obviously, as the immediate consequences of Theorem 1.1-1.3, we have the following

Corollary 1.1. Let $\delta \in [0, R)$ be given. Assume that (A1) and (A2) hold. Then there exists $\lambda_* \in (0, \lambda_1(m, \delta)]$ such that, for all $\lambda \in (0, \lambda_*)$, the problem $(1.5)_\delta$ has no positive solution and, for all $\lambda > \lambda_1(m, \delta)$ has at least one positive solution.

Corollary 1.2. Let $\delta \in [0, R)$ be given. Assume that (A1) and (A3) hold. Then the problem $(1.5)_\delta$ has at least one positive solution for any $\lambda > 0$.

Corollary 1.3. Let $\delta \in [0, R)$ be given. Assume that (A1) and (A4) hold. Then there exists $0 < \lambda_* \leq \lambda^*$ such that the problem $(1.5)_\delta$ has at least two positive radial solutions for $\lambda > \lambda^*$, while it has no positive solutions for $\lambda \in (0, \lambda_*)$.

Remark 1.1 Coelho et.al.[13] applied the global bifurcation technique to study $(1.5)_0$ in the case $N = 1$ in which the weight $m(\cdot)$ is allow to change sign. Coelho et.al.[14] applied the variational methods to obtain the existence and multiplicity of positive radial solutions of (1.4). However, they gave no information about the global structure of the set of positive radial solutions of (1.4). It is worth remarking that the study of global behavior of the positive radial solution curves is very useful for computing the numerical solution of (1.4) as it can be used to guide the numerical work. For example, it can be used to estimate the value of v in advance in applying the finite difference method, and it can be used to restrict the range of initial values we need to consider in applying the shooting method.

Remark 1.2 If $\delta \in (0, R)$, then Corollary 1.2 is new in the study of positive radial solutions of (1.4) in an annular domain. If $\delta = 0$, then Corollary 1.2 reduces to Theorem A.

Remark 1.3 If $\delta \in (0, R)$, then Corollary 1.3 is new in the study of positive radial solutions of (1.4) in an annular domain. If $\delta = 0$, then Corollary 1.3 partially generalizes the results of Theorem B in which

$$f(r, u) = \mu(r)u^q \quad \text{and} \quad \lambda_* = \lambda^*.$$

Remark 1.4 In [7, Section 3], Bereanu et.al. studied the problem

$$(r^{N-1}\phi_1(u'))' + \lambda r^{N-1}\mu(r)p(u) = 0, \quad u'(0) = u(R) = 0. \quad (1.7)$$

They proved (1.7) has at least one positive classical radial solution if

$$R^N < \lambda \left(\min_{r \in [0, R]} \mu(r) \right) \int_0^R (R-s)^N p(s) ds. \quad (1.8)$$

In particular, it is clear that the above condition is satisfied provided that λ is sufficiently large. Our Corollary 1.1-1.3 provide a value

$$\Lambda_0 := \max\{\lambda_*, \lambda_1(m, \delta)\},$$

which guarantee that (1.4) has a positive classical radial solution if $\lambda > \Lambda_0$.

The rest of the paper is organized as follows. In Section 2 we state some preliminary results on the superior limit of a sequence of connected components due to Luo and Ma[24]. Section 3 is devoted to establish the existence of connected component of radial positive solutions for the prescribed mean curvature problem in an annular domain via global bifurcation technique. Finally in Section 4, we shall use the components obtained in Section 3 to construct the desired components of radial positive solutions for the prescribed mean curvature problem in a ball and prove Theorem 1.1-1.3.

For other results concerning the problem associated to prescribed mean curvature equations in Minkowski space, we refer the reader to [5, 9, 20, 28].

2 Some notations and preliminary results

Let X be a Banach space with the norm $\|\cdot\|$. Let $M \subseteq X$ be a metric space and $\{C_n \mid n = 1, 2, \dots\}$ a family of subsets of M . Then the superior limit \mathcal{D} of $\{C_n\}$ is defined by

$$\mathcal{D} := \limsup_{n \rightarrow \infty} C_n = \{x \in M \mid \exists \{n_k\} \subset \mathbb{N}, x_{n_k} \in C_{n_k}, \text{ such that } x_{n_k} \rightarrow x\}. \quad (2.1)$$

A *component* of a set M means a maximal connected subset of M , see [32] for the detail.

For $\rho, \beta \in (0, \infty)$, let us denote $B_\rho := \{u \in X \mid \|u\| < \rho\}$ and $\Omega_{\beta, \rho} := ([0, \infty) \times X) \setminus \{(\mu, u) \in [\beta, \infty) \times X \mid \|u\| \leq \rho\}$.

The following results are somewhat scattered in Ma and An [25-26] and Ma and Gao [27]. The abstract version is given in Luo and Ma [24].

Lemma 2.1 ([26, Lemma 2.2]) Let X be a Banach space and let $\{C_n\}$ be a family of closed connected subsets of X . Assume that

- (i) there exist $z_n \in C_n$, $n = 1, 2, \dots$ and $z_* \in X$ such that $z_n \rightarrow z_*$;
- (ii) $\lim_{n \rightarrow \infty} r_n = \lim_{n \rightarrow \infty} \sup\{\|u\| \mid u \in C_n\} = \infty$;
- (iii) for every $R > 0$, $(\bigcup_{n=1}^{\infty} C_n) \cap B_R$ is a relatively compact of X .

Then there exists an unbounded component \mathcal{C} in \mathcal{D} and $z_* \in \mathcal{C}$.

Lemma 2.2 ([24, Theorem 3]) Let X be a Banach space. Let $\{C_n\}$ be a family of connected subsets of $\mathbb{R} \times X$. Assume that

(C1) $C_n \cap ((-\infty, 0] \times X) = \emptyset$;

(C2) there exist $0 < \sigma < r < \infty$ and $b \in (0, \infty)$ such that

$$C_n \cap \{(\mu, u) \mid \mu \geq b - \sigma, r - \sigma \leq \|u\| \leq r + \sigma\} = \emptyset;$$

(C3) $\mu_k > a$ for all $k \in \mathbb{N}$, $\mu_k \rightarrow +\infty$ and C_n meets $(\mu_n, \mathbf{0})$ and infinity in $([0, \infty) \times X) \setminus \Omega_{b,r}$;

(C4) for every $R > 0$, $(\bigcup_{n=1}^{\infty} C_n) \cap B_R$ is a relatively compact of X .

Then there exists an unbounded component \mathcal{C} in \mathcal{D} such that

(a) both $\mathcal{C} \cap \Omega_{b,r}$ and $\mathcal{C} \cap (([a, \infty) \times X) \setminus \Omega_{b,r})$ are unbounded;

(b) $\mathcal{C} \cap \{(\mu, u) \mid \mu \geq b, \|u\| = r\} = \emptyset$. □

We start by considering the auxiliary problem

$$\begin{cases} -(r^{N-1}u')' = r^{N-1}h(r), & r \in (\delta, R) \text{ with } \delta > 0, \\ u'(\delta) = 0 = u(R) \end{cases} \quad (2.2)$$

for a given $h \in X_\delta$. The Green function of (2.2) for $N \geq 3$ is explicitly given by

$$K_\delta(t, s) = \begin{cases} \frac{1}{2-N} [R^{2-N} - t^{2-N}], & \delta \leq s \leq t \leq R, \\ \frac{1}{2-N} [R^{2-N} - s^{2-N}], & \delta \leq t \leq s \leq R. \end{cases} \quad (2.3)$$

and the Green function of (2.2) for $N = 2$ is explicitly given by

$$K_\delta(t, s) = \begin{cases} \ln \frac{R}{t}, & \delta \leq s \leq t \leq R, \\ \ln \frac{R}{s}, & \delta \leq t \leq s \leq R. \end{cases} \quad (2.4)$$

It is well-known that for every $h \in X_\delta$, (2.3) has a unique solution

$$u = \int_\delta^R K_\delta(t, s) s^{N-1} h(s) ds =: \mathcal{G}_\delta(h) \quad (2.5)$$

It is easy to check that $\mathcal{G}_\delta : X_\delta \rightarrow E_\delta$ is continuous and compact (see [3]).

Lemma 2.3 For $\epsilon \in (0, \frac{R-\delta}{4})$, there exists $\beta = \beta(\epsilon) > 0$ such that

$$K_\delta(t, s) \geq \beta K_\delta(s, s), \quad (t, s) \in [\delta, R - \epsilon] \times [\delta, R]. \quad (2.6)$$

Lemma 2.4 Let

$$I_\delta(t) := \int_\delta^{\frac{R-\delta}{2}} K_\delta(t, s) s^{N-1} ds, \quad t \in [\delta, R]. \quad (2.7)$$

Then

(1) For the case $N \geq 3$,

$$I_\delta(t) := \frac{1}{2-N} \left[(R^{2-N} - t^{2-N}) \frac{t^N - \delta^N}{N} + R^{2-N} \frac{(\frac{R-\delta}{2})^N - t^N}{N} - \frac{(\frac{R-\delta}{2})^2 - t^2}{2} \right]; \quad (2.8)$$

$$I_0(t) := \frac{1}{2-N} \left[\left(\frac{1}{N 2^N} - \frac{1}{8} \right) R^2 + \left(\frac{1}{2} - \frac{1}{N} \right) t^2 \right] > 0, \quad t \in [0, R/2]; \quad (2.9)$$

$$\max_{0 \leq t \leq R/2} I_0(t) = \frac{1}{2-N} \left(\frac{1}{N 2^N} - \frac{1}{8} \right) R^2 > 0. \quad (2.10)$$

(2) For the case $N = 2$,

$$I_\delta(t) := -\frac{\delta^2}{2} \ln \frac{R}{t} - \frac{t^2}{4} + \left(\frac{R-\delta}{2} \right)^2 \left(\frac{1}{4} + \frac{1}{2} \ln \frac{2R}{R-\delta} \right); \quad (2.11)$$

$$I_0(t) := -\frac{t^2}{4} + \left(\frac{R}{2} \right)^2 \left(\frac{1}{4} + \frac{1}{2} \ln 2 \right) > 0, \quad t \in [0, R/2]; \quad (2.12)$$

$$\max_{0 \leq t \leq R/2} I_0(t) = \left(\frac{R}{2} \right)^2 \left(\frac{1}{4} + \frac{1}{2} \ln 2 \right) > 0. \quad (2.13)$$

3 Radial solutions for the prescribed mean curvature problem in an annular domain

Let $\delta \in (0, R)$ be a given constant in this section.

Let us consider the boundary value problem

$$\begin{aligned} \operatorname{div}(\phi_N(\nabla v)) + \lambda f(|x|, v) &= 0 \quad \text{in } \mathcal{A}, \\ \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \Gamma_1, \quad v = 0 \quad \text{on } \Gamma_2, \end{aligned} \quad (3.1)$$

where

$$\mathcal{A} = \{x \in \mathbb{R}^N : \delta < |x| < R\},$$

$$\Gamma_1 = \{x \in \mathbb{R}^N : |x| = \delta\}, \quad \Gamma_2 = \{x \in \mathbb{R}^N : |x| = R\},$$

$\frac{\partial v}{\partial \nu}$ and $|\cdot|$ denote the outward normal derivative of v and the Euclidean norm in \mathbb{R}^N , respectively.

Setting, as usual, $|x| = r$ and $v(x) = u(r)$, the above problem (3.1) reduces to

$$\begin{aligned} -(r^{N-1} \phi_1(u'))' &= \lambda r^{N-1} f(r, u), \\ u'(\delta) &= 0 = u(R). \end{aligned} \quad (3.2)_\delta$$

It is easy to check that to find a positive radial solution of (3.1), it is enough to find a positive solution of (3.2) $_\delta$.

Remark 3.1 It is worth remarking that (3.2) $_\delta$ is equivalent to

$$\begin{cases} -(r^{N-1} u')' = \lambda r^{N-1} [f(r, u) h(u') - \frac{N-1}{r} u^3], & r \in (\delta, R), \\ u'(\delta) = 0 = u(R). \end{cases} \quad (3.3)_\delta$$

Since the nonlinearity $F(r, u, p) := f(r, u) h(p) - \frac{N-1}{r} p^3$ is singular at $r = 0$ when $\delta = 0$, we cannot deal with (3.3) $_0$ via the spectrum of (1.6) $_0$ directly. However, $F(r, u, p)$ is regular at $r = \delta$ if $\delta > 0$, in this case, (3.3) $_\delta$ with $\delta > 0$ can be treated via the spectrum of (1.6) $_\delta$ and

the standard bifurcation technique. This is why we firstly study the prescribed mean curvature problem in an annular domain.

Lemma 3.1 [7, Lemma 1] Assume (A1) hold. Let u be a nontrivial solution of

$$-(r^{N-1}\phi_1(u'))' = \lambda r^{N-1}f(r, |u|), \quad u'(\delta) = 0 = u(R).$$

Then $u > 0$ on $[\delta, R)$ and u is strictly decreasing.

Lemma 3.2 Let $w_n \in E_\delta$ be decreasing for each $n \in N$. If

$$\lim_{n \rightarrow \infty} \|w_n\|_{C[\delta, R]} = 0,$$

then $w'_n \rightarrow 0$ in measure as $n \rightarrow \infty$.

Proof. Since $w_n(\delta) = \|w_n\|_{C[\delta, R]}$, it follows that

$$\lim_{n \rightarrow \infty} w_n(\delta) = 0.$$

For any $\bar{\sigma} > 0$, let

$$A_n(\bar{\sigma}) = \{x \in [\delta, R] : |w'_n(x) - 0| \geq \bar{\sigma}\}.$$

Then

$$w_n(\delta) = \int_\delta^R (-w'_n(x))dx = \int_\delta^R |w'_n(x) - 0|dx \geq \int_{A_n(\bar{\sigma})} |w'_n(x) - 0|dx \geq \bar{\sigma} \text{meas} A_n(\bar{\sigma}),$$

which means that $\text{meas} A_n(\bar{\sigma}) \rightarrow 0$. Therefore, $w'_n \rightarrow 0$ in measure. \square

3.1 Eigenvalue problem in an annular domain

Let $\delta \in (0, R)$ be given. Let us recall the weighted eigenvalue problem

$$\begin{cases} -(r^{N-1}u')' = \lambda r^{N-1}m(r)u, & r \in (\delta, R), \\ u'(\delta) = 0 = u(R), \end{cases} \quad (3.4)_\delta$$

where

(A5) $m \in C[\delta, R]$ and $m(r) \geq 0, m(r) \not\equiv 0$ on any subinterval of $[\delta, R]$.

The following result is a special case of [29, Theorem 1.5.3] when $p = 2$.

Lemma 3.3 Let (A5) hold. Then the eigenvalue problem $(3.4)_\delta$ has infinitely many simple real eigenvalues

$$0 < \lambda_1(m, \delta) < \lambda_2(m, \delta) < \cdots < \lambda_k(m, \delta) < \cdots \rightarrow +\infty \text{ as } k \rightarrow +\infty$$

and no other eigenvalues. Moreover, the algebraic multiplicity of $\lambda_k(m, \delta)$ is 1, and the eigenfunction φ_k corresponding to $\lambda_k(m, \delta)$ has exactly $k - 1$ simple zeros in (δ, R) .

Define a linear operator $\mathcal{L}_\delta : X_\delta \rightarrow E_\delta$ ($\hookrightarrow X_\delta$).

$$\mathcal{L}_\delta(u)(r) := \mathcal{G}_\delta(mu)(r).$$

Then \mathcal{L}_δ is compact and $(3.4)_\delta$ is equivalent to

$$u = \lambda \mathcal{L}_\delta(u). \quad (3.5)_\delta$$

Moreover, $\mathcal{L}_\delta|_{E_\delta} : E_\delta \rightarrow E_\delta$ is compact.

3.2 An equivalent formulation

Let us define a function $\tilde{f} : [\delta, R] \times \mathbb{R} \rightarrow \mathbb{R}$ by setting, for $r \in [\delta, R]$,

$$\tilde{f}(r, s) = \begin{cases} f(r, s), & \text{if } 0 \leq s \leq R - \delta, \\ 0, & \text{if } s \geq (R - \delta) + 1, \\ \text{linear}, & \text{if } R - \delta < s < (R - \delta) + 1, \\ -\tilde{f}(r, -s), & \text{if } s < 0. \end{cases}$$

Observe that, within the context of positive solutions, problem $(3.2)_\delta$ is equivalent to the same problem with f replaced by \tilde{f} . Indeed, if u is a positive solution, then $\|u'\|_{C[\delta, R]} < 1$ and hence $\|u\|_{C[\delta, R]} < R - \delta$. Clearly, \tilde{f} satisfies all the properties assumed in the statement of the theorem. Furthermore, $\tilde{f}(r, \cdot)$ is an odd function for $r \in [\delta, R]$. In the sequel, we shall replace f with \tilde{f} ; however, for the sake of simplicity, the modified function \tilde{f} will still be denoted by f . Next, let us define $h : \mathbb{R} \rightarrow \mathbb{R}$ by setting

$$h(y) = \begin{cases} (1 - y^2)^{\frac{3}{2}}, & \text{if } |y| \leq 1, \\ 0, & \text{if } |y| > 1. \end{cases} \quad (3.6)$$

Claim. A function $u \in C^1[\delta, R]$ is a positive solution of $(3.2)_\delta$ if and only if it is a positive solution of the problem

$$\begin{cases} -(r^{N-1}u')' = \lambda r^{N-1}f(r, u)h(u') - (N-1)r^{N-2}u'^3, & r \in (\delta, R), \\ u'(\delta) = 0 = u(R). \end{cases} \quad (3.7)_\delta$$

It is clear that a positive solution $u \in C^1[\delta, R]$ of $(3.2)_\delta$ is a positive solution of $(3.7)_\delta$ as well. Conversely, suppose that $u \in C^1[\delta, R]$ is a positive solution of $(3.7)_\delta$. We aim to show that

$$\|u'\|_{C[\delta, R]} < 1. \quad (3.8)$$

Assume by contradiction that this is not the case. Then we can easily find an interval $[a, b] \subseteq [\delta, R]$ such that, either $u'(a) = 0$, $0 < |u'(r)| < 1$ in (a, b) and $|u'(b)| = 1$, or $|u'(a)| = 1$, $0 <$

$|u'(r)| < 1$ in (a, b) and $u'(b) = 0$. Suppose the former case occurs (in the latter one the argument would be similar). The function u satisfies the equation

$$-\left(r^{N-1} \frac{u'}{\sqrt{1-u'^2}}\right)' = \lambda r^{N-1} f(r, u)$$

in $[a, b)$. For each $r \in (a, b)$, integrating over the interval $[a, r]$ and using (A1), we obtain

$$|\phi_1(u'(r))| = \left| \frac{1}{r^{N-1}} \int_a^r \lambda t^{N-1} f(t, u) dt \right| \leq M$$

and hence

$$|u'(r)| \leq \phi_1^{-1}(M)$$

for every $r \in [a, b)$. Since $\phi_1^{-1}(M) < 1$, taking the limit as $r \rightarrow b^-$ we obtain the contradiction $|u'(b)| < 1$. Therefore $\|u'\|_{C[\delta, R]} < 1$ and, as a consequence, u is a positive solution of $(3.2)_\delta$.

3.3 Proof of Theorem 1.1-1.3 with $\delta \in (0, R)$

In this subsection, we shall prove Theorem 1.1-1.3 in the case $\delta > 0$.

Proof of Theorem 1.1 with $\delta \in (0, R)$. By (A1) and (A2) we can write, for any $r \in [\delta, R]$ and every $s \in \mathbb{R}$,

$$f(r, s) = (m(r) + l(r, s))s,$$

where $l : [\delta, R] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and

$$\lim_{s \rightarrow 0} l(r, s) = 0 \tag{3.9}$$

uniformly in $[\delta, R]$. Let us set, for convenience, $k(y) = h(y) - 1$ for $y \in \mathbb{R}$. We have

$$\lim_{y \rightarrow 0} \frac{k(y)}{y} = 0. \tag{3.10}$$

Define the operator $\mathcal{H} : \mathbb{R} \times E_\delta \rightarrow E_\delta$ by

$$\mathcal{H}_\delta(\lambda, u) = \mathcal{G}_\delta(\lambda[l(\cdot, u) + (m + l(\cdot, u))k(u')])u - \gamma(\cdot)u'^3$$

where $\gamma(r) = \frac{N-1}{r}$. Clearly, \mathcal{H}_δ is completely continuous and, by (3.9) and (3.10),

$$\lim_{\|u\|_{C^1[\delta, R]} \rightarrow 0} \frac{\|\mathcal{H}_\delta(\lambda, u)\|_{C^1[\delta, R]}}{\|u\|_{C^1[\delta, R]}} = 0, \tag{3.11}$$

uniformly with respect to λ varying in bounded intervals. Observe that, for any λ , the couple $(\lambda, u) \in \mathbb{R} \times E_\delta$ is a positive solution of the equation

$$u = \lambda \mathcal{L}_\delta(u) + \mathcal{H}_\delta(\lambda, u) \tag{3.12}$$

if and only if u is a positive solution of $(3.2)_\delta$.

Recall that $\Sigma_\delta \subset \mathbb{R} \times E_\delta$ be the closure of the set of all nontrivial solutions (λ, u) of (3.12) with $\lambda > 0$. Note that the set $\{u \in E_\delta \mid (\lambda, u) \in \Sigma_\delta\}$ is bounded in E_δ .

As the algebraic multiplicity of $\lambda_1(m, \delta)$ equals 1 [23], the local index of 0 as a fixed point of $\lambda\mathcal{L}_\delta$ changes sign as λ crosses $\lambda_1(m, \delta)$. Therefore, according to a revised version of [21, Theorem 6.2.1], there exists a component, denoted by $\mathfrak{C} \subset \Sigma_\delta$, emanating from $(\lambda_1(m, \delta), \theta)$.

Notice that the positive cone P_δ is not a normal cone in E_δ , so we can not directly use the unilateral global bifurcation theory of López-Gómez [21, Sections 6.4-6.5]. However, as pointed out to us by Cano-Casanova et al. [11, page 5910], except for the normality of P_δ , Eq. (3.12) also enjoys all the structural requirements for applying the unilateral global bifurcation theory of López-Gómez [21, Sections 6.4-6.5], and the lack of the normality of P_δ is far from being a difficulty if one uses the generalized Krein-Rutman theorem [22, Theorem 6.3.1], for which the normality of P_δ is not required, as it is in some classical versions of the Krein-Rutman theorem (e.g., Amann [2], Krein and Rutman[19]).

Moreover, thanks to the global alternative of Rabinowitz (e.g., [21, Corollary 6.3.2]), either \mathfrak{C} is unbounded in $\mathbb{R} \times E_\delta$, or $(\lambda_j(m, \delta), \theta) \in \mathfrak{C}$ for some $\lambda_j(m, \delta) \neq \lambda_1(m, \delta)$.

Although the unilateral bifurcation Theorems 1.27 and 1.40 of Rabinowitz [30] cannot be applied here, among other things because they are false as originally stated (cf. the counterexample of Dancer [17]), the reflection argument of [30] can be applied to conclude that

$$\mathfrak{C} = \mathfrak{C}_+ \cup \mathfrak{C}_-,$$

where \mathfrak{C}_+ stands for the component of positive solutions emanating at $\lambda_1(m, \delta)$, as

$$\lambda\mathcal{L}_\delta(-u) + \mathcal{H}_\delta(\lambda, -u) = -[\lambda\mathcal{L}_\delta(u) + \mathcal{H}_\delta(\lambda, u)] \quad \forall u \in E_\delta.$$

Consequently, \mathfrak{C}_+ must be unbounded and, due to Lemma 3.1, $\mathfrak{C}_+ \subset (0, \infty) \times \text{int } P_\delta$.

Take

$$\zeta := \mathfrak{C}^+.$$

Obviously, (a) is true.

(b) can be deduced from the fact that

$$\sup\{\|u'\|_{C[\delta, R]} : (\lambda, u) \in \zeta\} \leq 1, \quad \sup\{\|u\|_{C[\delta, R]} : (\lambda, u) \in \zeta\} \leq R - \delta.$$

(c) Let

$$\lambda_* := \inf\{\lambda : (\lambda, u) \in \zeta\}.$$

We claim that $\lambda_* \in (0, \infty)$.

Suppose on the contrary that $\lambda_* = 0$. Then there exists a sequence $\{(\mu_n, u_n)\} \subset \zeta$ satisfying $u_n > 0$, and

$$\lim_{n \rightarrow \infty} (\mu_n, u_n) = (0, u^*) \quad \text{in } \mathbb{R} \times X_\delta$$

for some $u^* \geq 0$. Then it follows from

$$-(r^{N-1}\phi_1(u'_n))' = \mu_n r^{N-1} f(r, u_n), \quad u'_n(\delta) = 0 = u_n(R)$$

that, after taking a subsequence and relabeling, if necessary ,

$$u_n \rightarrow 0.$$

On the other hand,

$$\begin{cases} -(r^{N-1}u'_n)' = \mu_n r^{N-1} f(r, u_n) h(u'_n) - (N-1)r^{N-2}u_n'^3, & r \in (\delta, R), \\ u'_n(\delta) = 0 = u_n(R). \end{cases}$$

Setting, for all n , $v_n = u_n/||u_n||_{C[\delta, R]}$, we have that

$$\begin{cases} -(r^{N-1}v'_n)' = \mu_n r^{N-1} \frac{f(r, u_n)}{u_n} h(u'_n) v_n - (N-1)r^{N-2}u_n'^2 v'_n, & r \in (\delta, R), \\ v'_n(\delta) = 0 = v_n(R). \end{cases} \quad (3.13)$$

Notice that

$$r^{N-1}\phi(u'_n(r)) = -\mu_n \int_\delta^r \tau^{N-1} f(\tau, u_n(\tau)) d\tau, \quad r \in [\delta, R].$$

This together with $f(r, 0) = 0$ for $r \in [\delta, R]$ imply that

$$\lim_{n \rightarrow \infty} ||u'_n||_{C[\delta, R]} = 0.$$

Combining this with (3.13) and the facts $f_0 = m(r)$, $u_n \rightarrow 0$ and $\lim_{n \rightarrow \infty} h(u'_n) = 1$, it concludes that $\mu_n \rightarrow \lambda_1(m, \delta)$. This is a contradiction.

(d) We divide the proof into several steps.

Step 1 We claim that there exists two constants $B_0 > 0$ and $\rho_* > 0$, such that

$$||u||_{C[\delta, R]} \geq \rho_*, \quad (\lambda, u) \in \zeta \text{ with } \lambda \geq B_0.$$

Suppose on the contrary that there exists a sequence $(\mu_n, u_n) \in \zeta$ satisfying

$$(\mu_n, u_n) \rightarrow (\infty, \theta) \quad \text{in } (0, \infty) \times X_\delta.$$

Then from Lemma 3.2, it deduces u'_n converges to 0 in measure as $n \rightarrow \infty$. Combining this with the fact $u_n \rightarrow 0$ and using (3.13), it follows that, after taking a subsequence and relabeling, if necessary, $v_n \rightarrow v^*$ in X_δ for some $v^* \in X_\delta$, and furthermore,

$$-(r^{N-1}v'^*)' = \lambda_1(m, \delta) r^{N-1} m(r) v^*, \quad a.e. \ r \in (\delta, R), \quad v'^*(\delta) = 0 = v^*(R).$$

This contradicts with the fact $\mu_n \rightarrow \infty$. Therefore, the claim is true.

Step 2 We show that for arbitrary fixed $\epsilon \in (0, \frac{R-\delta}{4})$, there exists $\beta > 0$ such that for $(\lambda, u) \in \xi$ with λ_0 , we have

$$\min_{r \in [\delta, R-\epsilon]} u(r) \geq \beta \rho_*.$$

It is an immediate consequence of Lemma 2.3 and the fact

$$u(r) = \lambda \int_{\delta}^R K_{\delta}(r, s) s^{N-1} [f(s, u_n) h(u') - \frac{N-1}{s} u'^3] ds.$$

Step 3 We show that for every $n \in \mathbb{N}$, one has

$$\lim_{(\lambda, u) \in \xi, \lambda \rightarrow \infty} \text{meas}\{r \in [\delta, R] : |u'(r) - (-1)| > \frac{1}{n}\} = 0.$$

Since $\min_{r \in [\delta, R-\epsilon]} u(r) \geq \beta \rho_*$ and $f(r, s) > 0$ for $(r, s) \in [\delta, R] \times (0, \alpha)$, it follows that

$$f(s, u(s)) \geq M_0 > 0$$

for some constant $M_0 > 0$, and subsequently

$$\lim_{\lambda \rightarrow \infty} \lambda r^{1-N} \int_{\delta}^r s^{N-1} f(s, u(s)) ds = +\infty, \quad \text{uniformly in } r \in [\delta + \epsilon_1, R - \epsilon]$$

for arbitrary fixed $\epsilon_1 \in (0, \frac{R-\epsilon-\delta}{4})$. This together with relation

$$u'(r) = -(\phi_1)^{-1} (\lambda r^{1-N} \int_{\delta}^r s^{N-1} f(s, u(s)) ds)$$

imply that

$$u' \rightarrow -1 \quad \text{in } C[\delta + \epsilon_1, R - \epsilon], \quad \text{as } \lambda \rightarrow +\infty. \quad (3.14)$$

Therefore, by the arbitrariness of ϵ and ϵ_1 , we may get the desired result.

(e) From Lemma 3.1, we know that

$$-u'(r) \geq 0 \quad r \in (\delta, R],$$

This together with (3.14) imply that for $(\lambda, u) \in \xi$,

$$\lim_{\lambda \rightarrow \infty} \|u\|_{C[\delta, R]} = \lim_{\lambda \rightarrow \infty} u(\delta) = \lim_{\lambda \rightarrow \infty} \int_{\delta}^R [-u'(s)] ds \geq \lim_{\lambda \rightarrow \infty} \int_{\delta+\epsilon_1}^{R-\epsilon} [-u'(s)] ds = (R - \delta - \epsilon - \epsilon_1).$$

By the arbitrariness of ϵ and ϵ_1 and using the fact

$$u(\delta) = \int_{\delta}^R (-u'(s)) ds \leq R - \delta,$$

it concludes that

$$\lim_{\lambda \rightarrow \infty} \|u\|_{C[\delta, R]} = R - \delta.$$

□

In the following, we will deal with the cases that $f_0 = \infty$ and $f_0 = 0$, respectively.

Define $f^{[n]} : [\delta, R] \times \mathbb{R} \rightarrow \mathbb{R}$ as follows

$$f^{[n]}(r, s) = \begin{cases} nf(r, \frac{1}{n})s, & \text{if } s \in [0, \frac{1}{n}], \\ f(r, s), & \text{if } s \in (\frac{1}{n}, \infty), \\ -f^{[n]}(r, -s), & \text{if } s < 0. \end{cases}$$

Then $f^{[n]}$ is an odd function and satisfies (A1) and

$$(f^{[n]})_0 = nf(r, \frac{1}{n}) = f(r, \frac{1}{n})/(1/n) =: m^{[n]}(r) \quad \text{uniformly for } r \in [\delta, R].$$

Now, let us consider the auxiliary family of the problems

$$\begin{cases} -(r^{N-1}u')' = \lambda r^{N-1}f^{[n]}(r, u)h(u') - (N-1)r^{N-2}u'^3, & r \in (\delta, R), \\ u'(\delta) = 0 = u(R). \end{cases} \quad (3.15)$$

From the definition of $f^{[n]}$, it follows that for $r \in [\delta, R]$ and every $u \in \mathbb{R}$,

$$f^{[n]}(r, s) = (m^{[n]}(r) + \xi^{[n]}(r, s))s,$$

where $\xi^{[n]} : [\delta, R] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and

$$\lim_{s \rightarrow 0} \xi^{[n]}(r, s) = 0 \quad \text{uniformly for } r \in [\delta, R]. \quad (3.16)$$

Let us set, for convenience, $k(v) = h(v) - 1$ for $v \in \mathbb{R}$. We have

$$\lim_{v \rightarrow 0} \frac{k(v)}{v} = 0. \quad (3.17)$$

Define the operator $\mathcal{H}_\delta^{[n]} : \mathbb{R} \times E_\delta \rightarrow E_\delta$ by

$$\mathcal{H}_\delta^{[n]}(\lambda, u) = \mathcal{G}_\delta \left(\lambda (\xi^{[n]}(\cdot, u) + [m^{[n]} + \xi^{[n]}(\cdot, u)]k(u'))u - \gamma(\cdot)u'^3 \right).$$

Clearly, $\mathcal{H}_\delta^{[n]}$ is completely continuous and by (3.16) and (3.17), it follows that

$$\lim_{\|u\|_{C^1[\delta, R]} \rightarrow 0} \frac{\|\mathcal{H}_\delta^{[n]}(\lambda, u)\|_{C^1[\delta, R]}}{\|u\|_{C^1[\delta, R]}} = 0$$

uniformly with respect to λ varying in bounded intervals. Observe that, for any λ , the couple $(\lambda, u) \in \mathbb{R} \times E_\delta$ with $u > 0$, is a solution of the equation

$$u = \lambda \mathcal{L}_\delta^{[n]}(u) + \mathcal{H}_\delta^{[n]}(\lambda, u) \quad (3.18)$$

if and only if u is a positive solution of (3.15). Here $\mathcal{L}_\delta^{[n]} : X_\delta \rightarrow E_\delta$ be defined by $\mathcal{L}_\delta^{[n]}(u) = \mathcal{G}_\delta(m^{[n]}u)$.

Let $\Sigma_\delta^{[n]} \subset \mathbb{R} \times E_\delta$ be the closure of the set of all nontrivial solutions (λ, u) of (3.18) with $\lambda > 0$. Note that the set $\{u \in E_\delta | (\lambda, u) \in \Sigma_\delta^{[n]}\}$ is bounded in E_δ .

Remark 3.2. Note that from the compactness of the embedding $E_\delta \hookrightarrow X_\delta$, it concludes that $\mathfrak{C}_+^{[n]}$ is also an unbounded connected component in $[0, \infty) \times X_\delta$.

Proof of Theorem 1.2 with $\delta \in (0, R)$. Similar to the proof of Theorem 1.1 with $\delta \in (0, R)$, for each fixed n , there exists an unbounded component $\mathfrak{C}_+^{[n]} \subset \Sigma_\delta^{[n]}$ of positive solutions of (3.18) joining $(\lambda_1(m^{[n]}, \delta), \theta) \in \mathfrak{C}_+^{[n]}$ to infinity in $[0, \infty) \times P_\delta^0$. Moreover, $(\lambda_1(m^{[n]}, \delta), \theta) \in \mathfrak{C}_+^{[n]}$ is the only positive bifurcation point of (3.18) lying on a trivial solution line $u \equiv \theta$ and the component $\mathfrak{C}_+^{[n]}$ joins the infinity in the direction of λ since u is bounded.

It is not difficult to verify that $\mathfrak{C}_+^{[n]}$ satisfies all conditions in Lemma 2.1 and consequently $\limsup_{n \rightarrow \infty} \mathfrak{C}_+^{[n]}$ contains a component \mathfrak{C}_+ which is unbounded.

From (A3), it follows that for $r \in [\delta, R]$,

$$\lim_{n \rightarrow \infty} \frac{f^{[n]}(r, u)}{u} = \lim_{n \rightarrow \infty} \frac{f(r, \frac{1}{n})}{1/n} = \infty,$$

and consequently,

$$\lim_{n \rightarrow \infty} \lambda_1(m^{[n]}, \delta) = 0. \quad (3.19)$$

Thus, from (3.19), we have that the component \mathfrak{C}_+ joins $(0, \theta)$ with infinity in the direction of λ in $[0, \infty) \times P_\delta^0$.

We claim that

$$(\mathfrak{C}_+ \setminus \{(0, \theta)\}) \subset (0, \infty) \times \text{int } P_\delta^0. \quad (3.20)$$

Suppose on the contrary that there exists a sequence $\{(\mu_n, u_n)\} \subset \mathfrak{C}_+$ satisfying $u_n > 0$, and

$$\lim_{n \rightarrow \infty} (\mu_n, u_n) = (\mu^*, \theta) \quad \text{in } \mathbb{R} \times X_\delta$$

for some $\mu^* > 0$. Then

$$\begin{cases} -(r^{N-1}u_n')' = \mu_n r^{N-1} f^{[n]}(r, u_n) h(u_n') - (N-1)r^{N-2}u_n'^3, & r \in (\delta, R), \\ u_n'(\delta) = 0 = u_n(R). \end{cases}$$

Setting, for all n , $v_n = u_n/||u_n||_{C[\delta, R]}$, we have that

$$\begin{cases} -(r^{N-1}v_n')' = \mu_n r^{N-1} \frac{f^{[n]}(r, u_n)}{u_n} h(u_n') v_n - (N-1)r^{N-2}u_n'^2 v_n', & r \in (\delta, R), \\ v_n'(\delta) = 0 = v_n(R). \end{cases} \quad (3.21)$$

Notice that

$$r^{N-1}\phi(u_n'(r)) = -\mu_n \int_0^r \tau^{N-1} f^{[n]}(\tau, u_n(\tau)) d\tau, \quad r \in [0, R]. \quad (3.22)$$

This together with $f^{[n]}(r, 0) = 0$ for $r \in [\delta, R]$ imply that

$$\lim_{n \rightarrow \infty} ||u_n'||_{C[0, R]} = 0. \quad (3.23)$$

Combining this with (3.21) and the facts $f_0 = \infty$ and $\lim_{n \rightarrow \infty} h(u_n') = 1$, it concludes that $\mu^* = 0$. This is a contradiction.

Therefore, due to Lemma 3.1, (3.20) holds. \square

Proof of Theorem 1.3 with $\delta \in (0, R)$. Similar to the proof of Theorem 1.1, for each fixed n , there exists an unbounded component $\mathfrak{C}_+^{[n]} \subset \Sigma_\delta^{[n]}$ of positive solutions of (3.18) joining $(\lambda_1(m^{[n]}, \delta), \theta) \in \mathfrak{C}_+^{[n]}$ to infinity in $[0, \infty) \times P_\delta^0$. Moreover, $(\lambda_1(m^{[n]}, \delta), \theta) \in \mathfrak{C}_+^{[n]}$ is the only positive bifurcation point of (3.18) lying on a trivial solution line $u \equiv \theta$ and the component $\mathfrak{C}_+^{[n]}$ joins the infinity in the direction of λ since u is bounded.

From (A4) it follows that for $r \in [\delta, R]$ and every $u \in (0, \frac{1}{n}]$,

$$\lim_{n \rightarrow \infty} \frac{f^{[n]}(r, u)}{u} = \lim_{n \rightarrow \infty} \frac{f(r, 1/n)}{1/n} = 0,$$

and consequently

$$\lim_{n \rightarrow \infty} \lambda_1(m^{[n]}, \delta) = \infty.$$

We claim that there exists $\Lambda_\delta \in (0, \infty)$, such that for each n ,

$$\mathfrak{C}_+^{[n]} \cap \{(\mu, v) \in \Sigma_\delta \mid \mu \geq \Lambda_\delta, \rho_0 - \frac{\rho_0}{8} \leq \|v\|_{C[\delta, R]} \leq \rho_0 + \frac{\rho_0}{8}\} = \emptyset, \quad (3.24)$$

where $\rho_0 := \frac{R-\delta}{4}$.

In fact, if $(\lambda, u) \in \mathfrak{C}_+^{[n]}$ is a solution with

$$\rho_0 - \frac{\rho_0}{8} \leq \|u\|_{C[\delta, R]} \leq \rho_0 + \frac{\rho_0}{8}.$$

Let $N_* \in \mathbb{N}$ be an integer such that

$$\frac{1}{N_*} < \beta \rho_0.$$

Then, for $n \geq N_*$, we have

$$f^{[n]}(r, s) = f(r, s), \quad (r, s) \in [\delta, R] \times [\beta \rho_0, \infty).$$

Denote

$$I_1 = \{s \in [\delta, \frac{R-\delta}{2}] : |u'(s)| \leq \frac{1}{2}\}, \quad I_2 = \{s \in [\delta, \frac{R-\delta}{2}] : |u'(s)| > \frac{1}{2}\}.$$

Thus

$$\begin{aligned} \frac{9}{8}\rho_0 &= \|u\|_{C[\delta, R]} \\ &= \lambda \max_{\delta \leq r \leq R} \int_\delta^R K_\delta(r, s) s^{N-1} [f^{[n]}(s, u) h(u') - \frac{N-1}{s} u'^3] ds \\ &\geq \lambda \max_{\delta \leq r \leq R} \int_\delta^{\frac{R-\delta}{2}} K_\delta(r, s) s^{N-1} [f^{[n]}(s, u) h(u') - \frac{N-1}{s} u'^3] ds \end{aligned}$$

$$\begin{aligned}
&\geq \lambda \max_{\delta \leq r \leq R} \left(\int_{I_1} K_\delta(r, s) s^{N-1} [f^{[n]}(s, u) h(u')] ds - \int_{I_2} K_\delta(r, s) s^{N-1} \left[\frac{N-1}{s} u'^3 \right] ds \right) \\
&\geq \lambda \max_{\delta \leq r \leq R} \left(\int_{I_1} K_\delta(r, s) s^{N-1} [f^{[n]}(s, u) \frac{1}{2}] ds + \int_{I_2} K_\delta(r, s) s^{N-1} \left[\frac{N-1}{s} \left(\frac{1}{2} \right)^3 \right] ds \right) \\
&\geq \lambda \max_{\delta \leq r \leq R} \left(\int_{I_1} K_\delta(r, s) s^{N-1} [f(s, u) \frac{1}{2}] ds + \int_{I_2} K_\delta(r, s) s^{N-1} \left[\frac{N-1}{s} \left(\frac{1}{2} \right)^3 \right] ds \right) \\
&\geq \lambda \min \left\{ \frac{m_f(\rho_0, \delta)}{2}, \frac{N-1}{8R} \right\} \max_{\delta \leq r \leq R} \int_{\delta}^{\frac{R-\delta}{2}} K_\delta(r, s) s^{N-1} ds, \\
&\geq \lambda \min \left\{ \frac{m_f(\rho_0, \delta)}{2}, \frac{N-1}{8R} \right\} \max_{\delta \leq r \leq R/2} \int_{\delta}^{\frac{R-\delta}{2}} K_\delta(r, s) s^{N-1} ds,
\end{aligned}$$

where

$$m_f(\rho_0, \delta) = \min \left\{ |f(r, u)| : r \in [\delta, R], \beta \rho_0 \leq u \leq \rho_0 \right\}.$$

Choose

$$\Lambda_\delta := \frac{9}{8} \rho_0 \left(\min \left\{ \frac{m_f(\rho_0, \delta)}{2}, \frac{N-1}{8R} \right\} \max_{\delta \leq r \leq R/2} \int_{\delta}^{\frac{R-\delta}{2}} K_\delta(r, s) s^{N-1} ds \right)^{-1} + \frac{1}{8} \rho_0. \quad (3.25)$$

Obviously, Λ_δ is independent of n , and (3.24) holds for all $\lambda > \Lambda_\delta$.

Now, by Lemma 2.2, there exist a connected component $\xi \in \Sigma_\delta$ and a constant $\Lambda_\delta > 0$, such that

- (i) ξ joins (∞, θ) with infinity in the direction of λ ;
- (ii) $\xi \cap \{(\mu, v) \in \Sigma_\delta \mid \mu \geq \Lambda_\delta, \|v\|_{C[\delta, R]} = \rho_0\} = \emptyset$.

Finally we show that

$$\text{Proj}_{\mathbb{R}} \xi = [\lambda_*, \infty) \subset (0, \infty)$$

for some $\lambda_* > 0$.

Suppose that there exists a sequence $\{(\mu_n, u_n)\}$ of nonnegative solutions of (3.15), converging in ξ to some $(0, u) \in \mathbb{R} \times E_\delta$. Arguing as in the proof of Claim (3.20), we set $v_n = \frac{u_n}{\|u_n\|_{C[\delta, R]}}$ and conclude that, possibly passing to a subsequence, $\lim_{n \rightarrow \infty} v_n = 0$ in E_δ , which contradicts $\|v_n\|_{C[\delta, R]} = 1$. Therefore, $\lambda_* > 0$. \square

4 Radial solutions for the prescribed mean curvature problem in a ball

In this section, we shall deal with $(1.5)_\delta$ with $\delta = 0$.

Let

$$g_n(r, s) = \begin{cases} 0, & (r, s) \in (0, \frac{1}{n}] \times (0, \alpha), \\ f(r - \frac{1}{n}, s), & (r, s) \in (\frac{1}{n}, R) \times (0, \alpha). \end{cases} \quad (4.1)$$

In the following, we shall use the positive solutions of the family of problems

$$\begin{aligned} -(r^{N-1}\phi_1(u'))' &= \lambda r^{N-1}g_n(r, u), \quad r \in (\frac{1}{n}, R), \\ u'(1/n) &= 0 = u(R) \end{aligned} \tag{4.2}_n$$

to construct the radial positive solutions of the prescribed mean curvature problem in a ball

$$\mathcal{M}v + \lambda f(|x|, v) = 0 \quad \text{in } \mathcal{B}(R), \quad v = 0 \quad \text{on } \partial\mathcal{B}(R). \tag{4.3}$$

To find a radial positive solution of (4.3), it is enough to find a positive solution of the problem

$$\begin{aligned} -(r^{N-1}\phi_1(u'))' &= \lambda r^{N-1}f(r, u), \\ u'(0) &= 0 = u(R). \end{aligned} \tag{4.4}$$

For given $n \in \mathbb{N}$, let (λ, u) be a positive solution of $(4.2)_n$. For each n , define a function $y_n : [0, R] \rightarrow [0, \infty)$ by

$$y_n(r) = \begin{cases} u(r), & \frac{1}{n} \leq r \leq R, \\ u(\frac{1}{n}), & 0 \leq r \leq \frac{1}{n}. \end{cases} \tag{4.5}$$

Then

$$y_n \in \{w \in C^2[0, R] : w'(0) = w(R) = 0\}.$$

Moreover, y_n is a positive solution of the problem

$$\begin{aligned} -(r^{N-1}\phi_1(u'))' &= \lambda r^{N-1}g_n(r, u), \quad r \in (0, R) \\ u'(0) &= 0 = u(R), \end{aligned} \tag{4.6}_n$$

i.e. y_n is a positive solution of the problem

$$\begin{aligned} -(r^{N-1}u'(r))' + (N-1)r^{N-2}[u'(r)]^3 &= \lambda r^{N-1}g_n(r, u(r))h(u'(r)), \quad r \in (0, R), \\ u'(0) &= u(R) = 0. \end{aligned} \tag{4.7}_n$$

On the other hand, if (λ, y) is a solution of $(4.7)_n$, then $(\lambda, y|_{[\frac{1}{n}, R]})$ is a solution of $(4.2)_n$.

Lemma 4.1 Let (A1) and (A2) hold. Let $\hat{\lambda} : \hat{\lambda} \neq \lambda_1(m, 0)$ be given. Then there exists $\hat{b} > 0$, such that

$$\|u\|_{C[0, R]} \geq \hat{b}$$

for any positive solution $(\hat{\lambda}, u)$ of $(4.7)_n$. Here b is independent of n and u .

Proof. Suppose on the contrary that $(4.7)_n$, $n \in \mathbb{N}$, has a sequence of positive solution $(\hat{\lambda}, y_j)$ with

$$\lim_{j \rightarrow \infty} \|y_j\|_{C[0, R]} = 0. \tag{4.8}$$

Then

$$\begin{aligned} (r^{N-1}\phi(y_j'(r)))' + \hat{\lambda}r^{N-1}g_n(r, y_j(r)) &= 0, \quad r \in (0, R), \\ y_j'(0) &= y_j(R) = 0, \end{aligned} \tag{4.9}$$

and consequently,

$$r^{N-1}\phi(y'_j(r)) = -\hat{\lambda} \int_0^r \tau^{N-1} g_n(\tau, y_j(\tau)) d\tau, \quad r \in [0, R].$$

This together with (4.8) and the fact that $g_n(r, 0) = 0$ for $r \in [0, R]$ imply that

$$\lim_{j \rightarrow \infty} \|y'_j\|_{C[0, R]} = 0. \quad (4.10)$$

Recall that (4.9) can be rewritten as

$$\begin{aligned} & -(r^{N-1}y'_j(r))' + (N-1)r^{N-2}[y'_j(r)]^3 = \hat{\lambda}r^{N-1}g_n(r, y_j(r))h(y'_j(r)), \\ & y'_j(0) = y_j(R) = 0. \end{aligned} \quad (4.11)_n$$

Setting, for all j , $v_j = y_j/(\|y_j\|_{C[0, R]})$, we have that

$$\begin{aligned} & -(r^{N-1}v'_j(r))' + (N-1)r^{N-2}[v'_j(r)]^2v'_j(r) = \hat{\lambda}r^{N-1}\frac{g_n(r, y_j(r))}{y_j(r)}v_j(r)h(y'_j(r)), \\ & v'_j(0) = v_j(R) = 0. \end{aligned} \quad (4.12)_n$$

Letting $j \rightarrow \infty$, it follows from (4.8), (4.10) and (4.12)_n that there exists $w \in C^2[0, R]$ with $\|w\|_{C[0, R]} = 1$ and $w > 0$ in $[0, R)$, such that

$$\begin{aligned} & -(r^{N-1}w(r))' = \hat{\lambda}r^{N-1}m(r)w(r), \\ & w'(0) = w(R) = 0, \end{aligned} \quad (4.13)$$

which implies that $\hat{\lambda} = \lambda_1(m, 0)$. However, this contradicts the assumption $\hat{\lambda} \neq \lambda_1(m, 0)$. \square

Using the same argument with obvious changes, we may prove the following

Lemma 4.2 Let (A1) and (A3) hold. Let $\hat{\lambda} \in (0, \infty)$ be given. Then there exists $\hat{b} > 0$, such that

$$\|u\|_{\infty} \geq \hat{b}$$

for any positive solution $(\hat{\lambda}, u)$ of (4.7)_n. \square

Lemma 4.3 Let (A1) and (A4) hold. Let $\hat{\lambda} \in (0, \infty)$ be such that (4.7)_n has a positive solutions for some n . Then there exists $\hat{b} > 0$, such that

$$\|u\|_{\infty} \geq \hat{b}$$

for any positive solution $(\hat{\lambda}, u)$ of (4.7)_n (if it has positive solution). \square

Now, we are in the position to prove Theorem 1.1-1.3 with $\delta = 0$.

Proof of Theorem 1.1 with $\delta = 0$. For given n , let ξ_n be the component obtained by Theorem 1.1 with $\delta \in (0, R)$ for (4.2)_n. Let

$$\zeta_n := \{(\lambda, y_n) : y_n \text{ is determined by } u \text{ via (4.5)}_n \text{ for } (\lambda, u) \in \xi_n\}.$$

Then ζ_n is a component in $[0, \infty) \times C^1[0, R]$ which joins $(\lambda_1(m^{[n]}, \frac{1}{n}), \theta)$ with infinity in the direction of λ and

$$\sup\{\|y\|_{C^1[0, R]} : (\lambda, y) \in \xi_n\} < M \quad (4.14)$$

for some constant $M > 0$, independent of y and n . Here

$$m^{[n]}(r) := m(r - \frac{1}{n}), \quad \frac{1}{n} \leq r \leq R,$$

and $\lambda_1(m^{[n]}, \frac{1}{n})$ is the principal eigenvalue of the linear problem

$$\begin{aligned} -(r^{N-1}u'(r))' &= \lambda r^{N-1}m^{[n]}(r)u(r), \quad r \in (\frac{1}{n}, R), \\ u'(\frac{1}{n}) &= u(R) = 0. \end{aligned} \quad (4.15)$$

Since $\lim_{n \rightarrow \infty} \lambda_1(m^{[n]}, \frac{1}{n}) = \lambda_1(m, 0)$, it follows from Lemma 2.1 that there exists a component ζ in $\limsup_{n \rightarrow \infty} \zeta_n$ which joins $(\lambda_1(m, 0), \theta)$ with infinity in the direction of λ and

$$\sup\{\|y\|_{C^1} : (\lambda, y) \in \zeta\} \leq M. \quad (4.16)$$

Now, Lemma 4.1 ensures that

$$\zeta \cap ([0, \infty) \times \{\theta\}) = \{(\lambda_1(m, 0), \theta)\}.$$

□

Proof of Theorem 1.2 with $\delta = 0$. It is an immediate consequence of Theorem 1.2 with $\delta > 0$ and Lemma 4.2. □

Proof of Theorem 1.3 with $\delta = 0$. For given n , let ξ_n be the component obtained by Theorem 1.3 with $\delta \in (0, R)$ for $(4.2)_n$, let Λ_n be the constant obtained in (3.25) for $(4.2)_n$, i.e.

$$\begin{aligned} \Lambda_{1/n} &= \frac{9R - 9/n}{32} \left(\min \left\{ \frac{m_f(\frac{R-1/n}{4}, 1/n)}{2}, \frac{N-1}{8R} \right\} \max_{1/n \leq r \leq R/2} \int_{1/n}^{\frac{R-1/n}{2}} K_{1/n}(r, s) s^{N-1} ds \right)^{-1} \\ &\quad + \frac{R - 1/n}{32}. \end{aligned} \quad (4.17)$$

Then

$$\xi^{[n]} \cap \{(\mu, v) \in \Sigma_{1/n} \mid \mu \geq \Lambda_{1/n}, \|v\|_{C[1/n, R]} = \frac{R - 1/n}{4}\} = \emptyset. \quad (4.18)$$

By Lemma 2.4, we may choose a constant

$$\Lambda_* := \frac{9R}{32} \left(\min \left\{ \frac{m_f(\frac{R}{4}, 0)}{2}, \frac{N-1}{8R} \right\} \max_{0 \leq r \leq R/2} I_0(r) \right)^{-1} + \frac{R}{32} + 1. \quad (4.19)$$

Then it is easy to see from (4.17) that there exists $N^* \in \mathbb{N}$, such that

$$\Lambda_{1/n} < \Lambda_*, \quad n \geq N^*. \quad (4.20)$$

Now, let ξ be the connected component in $\limsup \xi_n$ obtained Lemma 2.2. Then ξ joins (∞, θ) with infinity in $(0, \infty) \times \{z \in X_0 \mid \|z\|_{C[0,R]} \geq \frac{R}{4}\}$. Moreover, (4.18) and (4.20) yield that

$$\xi \cap \{(\mu, v) \in \Sigma_0 \mid \mu \geq \Lambda_*, \|v\|_{C[0,R]} = \frac{R}{4}\} = \emptyset.$$

□

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